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# The $SU(2)$ instanton and the adiabatic evolution of two Kramers doublets

M T Johnsson† and I J R Aitchison‡

† Department of Physics and Astronomy, University of Canterbury, Private Bag 4800, Christchurch, New Zealand

‡ Department of Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, UK

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**Abstract.** The adiabatic evolution of two doubly degenerate (Kramers) levels is considered. The general five-parameter Hamiltonian describing the system is obtained and is shown to be equivalent to one used in the $\Gamma_8 \otimes (\tau_2 \oplus \epsilon)$ Jahn–Teller system. It is shown explicitly that the resulting $SU(2)$ non-Abelian geometric vector potential is that of the ($SO(5)$ symmetric) $SU(2)$ instanton. Various forms of the potentials are discussed.

## 1. Introduction

Adiabatic evolution generates remarkable geometrical structures, as Berry [1] was the first to emphasize. The evolution of a single non-degenerate state is associated with a geometric $U(1)$ vector potential, which is a function of the adiabatically changing parameters $\boldsymbol{r}$. If this state becomes accidentally degenerate in energy with another state at some point $\boldsymbol{r}^*$ in parameter space, the $U(1)$ potential is that of a magnetic monopole situated at $\boldsymbol{r}^*$ [1]. When the Hamiltonian is restricted to be real (rather than Hermitian) the $U(1)$ potential is that of a flux tube [1, 2]. Examples of both situations are known in Jahn–Teller systems: the monopole in the $T \otimes \tau_2$ system [3], and the flux tube in the $E \otimes \epsilon$ system [4].

If the evolving state is itself degenerate throughout the evolution, the associated vector potential is non-Abelian [5, 6]. A natural question to ask then is the following. Suppose two such doublets become accidentally degenerate (four-fold degeneracy in all) at some point in parameter space: what will be the nature of the non-Abelian potentials? The answer to this question was, in fact, given some time ago [7]: namely the potentials are those of the $SU(2)$ Yang–Mills instanton [8–10]. However, the elegant mathematics of [7] did not descend to the explicit construction of the instanton potentials, which are the quantities most physicists prefer to deal with. Indeed, since such a degeneracy has co-dimension five (the geometric Hamiltonian depending on five parameters), the relationship of the five-dimensional potentials to those of the instanton, which is normally thought of as living in four-dimensional Euclidean space, is not completely self-evident. Finally, no specific physical example was considered in [7].

The purpose of the present paper is to fill these gaps. In section 2 we briefly recapitulate the case of a two-level crossing and the associated $U(1)$ monopole potential, in order to bring out later the very close analogy with the instanton. In section 3 we obtain the generic five-parameter Hamiltonian describing this degeneracy pattern, and observe that it is equivalent to that used in the $\Gamma_8 \otimes (\tau_2 \oplus \epsilon)$ Jahn–Teller system. In section 4 we

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calculate the associated geometric vector potentials in a five-dimensional Cartesian basis, and show—using the formalism of Jackiw and Rebbi [11]—how they are in fact identical to the familiar four-dimensional instanton potentials. In section 5 we adopt the coordinate system used by Yang [12] in his detailed study of the $SU(2)$ instanton (which he called a generalized monopole), and show once more that the adiabatically generated potentials agree exactly with Yang's.

## 2. The $U(1)$ monopole and two-level crossing

The generic Hermitian Hamiltonian for any system with accidental two-level crossing involves three parameters $\boldsymbol{r} = (r_1, r_2, r_3)$ and has the form

$$H = \boldsymbol{r}\cdot\boldsymbol{\sigma} \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. The eigenvalues of (1) are $\pm r$, where $r = |\boldsymbol{r}|$. One choice of normalized eigenvectors is

$$\psi_N^+ = \frac{1}{[2r(r+r_3)]^{\frac{1}{2}}}\begin{pmatrix} r+r_3 \\ r_1 + \mathrm{i}r_2\end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}\end{pmatrix} \tag{2}$$

corresponding to the eigenvalue $+r$, and

$$\psi_N^- = \frac{1}{[2r(r+r_3)]^{\frac{1}{2}}}\begin{pmatrix} -r_1+\mathrm{i}r_2 \\ r + r_3\end{pmatrix} = \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix} \tag{3}$$

corresponding to the eigenvalue $-r$. In (2) and (3) we have given the forms in both Cartesian coordinates $\boldsymbol{r} = (r_1, r_2, r_3)$ and in spherical polars $\boldsymbol{r} = (r, \theta, \phi)$.

The geometric vector potential $A_a$ is defined by $A_a = \langle\psi|\mathrm{i}\partial_a|\psi\rangle$, where the index $a$ runs over the number of parameters. In the present case, a short calculation gives

$$\boldsymbol{A}_N^{\pm} \equiv \langle\psi_N^{\pm}|\mathrm{i}\nabla|\psi_N^{\pm}\rangle = \frac{\mp 1}{2r(r+r_3)}(-r_2, r_1, 0) \tag{4}$$

or

$$(\boldsymbol{A}_N^{\pm})_\phi = \frac{\mp(1-\cos\theta)}{2r\sin\theta}. \tag{5}$$

Potentials (4) and (5) are those of a magnetic monopole of strength $\mp\frac{1}{2}$ [10, 13], located at the level-crossing point $r = 0$. The potentials $\boldsymbol{A}_N^{\pm}$ are evidently singular at $\theta = \pi$, and the corresponding eigenvectors are ill defined at that point. As is well known [14] this is a consequence of the fact that the potential for a monopole must be singular on at least one continuous line running from the monopole to infinity (the Dirac string). To avoid the singularity one can cover the sphere $S^2$ with two coordinate patches and define a non-singular vector potential in each patch. The potentials are linked by a gauge transformation in the region where the patches overlap. As the notation implies, in the present case the potentials $\boldsymbol{A}_N$ are non-singular over all the surface of $S^2$ except for the south pole $\theta = \pi$. Correspondingly, one can obtain potentials which are non-singular except at the north pole $\theta = 0$ by using the eigenvectors

$$\psi_S^+ = \frac{1}{[2r(r-r_3)]^{\frac{1}{2}}}\begin{pmatrix} r_1-\mathrm{i}r_2 \\ r - r_3\end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi}\cos\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix} \tag{6}$$

and

$$\psi_S^- = \frac{1}{[2r(r-r_3)]^{\frac{1}{2}}}\begin{pmatrix} -r+r_3 \\ r_1 + \mathrm{i}r_2\end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \mathrm{e}^{\mathrm{i}\phi}\cos\frac{\theta}{2}\end{pmatrix}. \tag{7}$$

The '$S$' potentials are (see also [13])

$$\boldsymbol{A}_S^{\pm} \equiv \langle \psi_S^{\pm} | \mathrm{i}\nabla | \psi_S^{\pm} \rangle = \frac{\mp 1}{2r(r - r_3)}(r_2, -r_1, 0) \tag{8}$$

or

$$(\boldsymbol{A}_N^{\pm})_\phi = \frac{\mp(-1 - \cos\theta)}{2r\sin\theta}. \tag{9}$$

$\boldsymbol{A}_N^{\pm}$ are therefore the potentials in the 'northern hemisphere' patch, and $\boldsymbol{A}_S^{\pm}$ those in the 'southern hemisphere' patch. From the spherical polar forms of (2), (3), (6) and (7), we see immediately that the $\psi_N^{\pm}$ are related to the $\psi_S^{\pm}$ by a phase transformation

$$\psi_N^{\pm} = \mathrm{e}^{\pm \mathrm{i}\phi} \psi_S^{\pm} \tag{10}$$

implying that $\boldsymbol{A}_N^{\pm}$ and $\boldsymbol{A}_S^{\pm}$ are related by a gauge transformation

$$\boldsymbol{A}_N^{\pm} - \boldsymbol{A}_S^{\pm} = \mp \nabla\phi \tag{11}$$

which is consistent with (5) and (9). If we take the equator $\theta = \pi/2$ as the overlap region between the $N$ and $S$ patches, we see that after a full circuit of the equator the geometrical phase $\exp[\mathrm{i}\oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r}]$ matches smoothly (via (10)) from $N$ to $S$, but the non-trivial nature of the gauge transformation (10) means that, in mathematical language, the $U(1)$ bundle over $S^2$ is non-trivial, and is indeed the monopole bundle.

## 3. The Hamiltonian for the crossing of two doublets

We require a situation in which the two doublets remain degenerate through adiabatic evolution. This can be ensured only by an appropriate symmetry, and the natural one to consider here is time-reversal symmetry. If a system is even under time-reversal and has half-odd integral total angular momentum, then each energy eigenstate will be at least doubly degenerate (Kramers degeneracy). We therefore consider a pair of levels each of which is a Kramers doublet, and construct the most general Hamiltonian, $H$, describing such a system.

The $4 \times 4$ matrix representation of $H$ must be Hermitian, and we choose a basis such that $H$ is traceless, making the two doublets degenerate at zero energy. Further, we let $T$ denote the time-reversal operator and $|\phi\rangle$, $|\bar{\phi}\rangle \equiv T|\phi\rangle$, $|\psi\rangle$, $|\bar{\psi}\rangle$ represent the two Kramers doublets where $T^2 = -1$ and $THT^{-1} = H$.

These equations lead to the relations

$$\langle\phi|H|\bar{\phi}\rangle = 0 \tag{12}$$

$$\langle\phi|H|\phi\rangle = \langle\bar{\phi}|H|\bar{\phi}\rangle \tag{13}$$

$$\langle\phi|H|\psi\rangle = \langle\bar{\phi}|H|\bar{\psi}\rangle^* \tag{14}$$

$$\langle\phi|H|\bar{\psi}\rangle = -\langle\bar{\phi}|H|\psi\rangle^*. \tag{15}$$

These constraints lead to a five-parameter description of the Hamiltonian in the basis $\{|\phi\rangle, |\bar{\phi}\rangle, |\psi\rangle, |\bar{\psi}\rangle\}$:

$$H = \begin{pmatrix} r_5 & 0 & r_3 + \mathrm{i}r_4 & r_1 + \mathrm{i}r_2 \\ 0 & r_5 & -r_1 + \mathrm{i}r_2 & r_3 - \mathrm{i}r_4 \\ r_3 - \mathrm{i}r_4 & -r_1 - \mathrm{i}r_2 & -r_5 & 0 \\ r_1 - \mathrm{i}r_2 & r_3 + \mathrm{i}r_4 & 0 & -r_5 \end{pmatrix}. \tag{16}$$

We note that this Hamiltonian can be identified with the Hamiltonian of [15] in their consideration of the $\Gamma_8 \otimes (\tau_2 \oplus \epsilon)$ Jahn–Teller system. To do this we interchange two of their basis states:

$$(|1\rangle, |2\rangle, |3\rangle, |4\rangle) \leftrightarrow (|1\rangle, |4\rangle, |3\rangle, |2\rangle) \tag{17}$$

and identify

$$\begin{aligned} r_1 &= V_T \cos\beta \sin\theta \cos\phi \\ r_2 &= -V_T \cos\beta \sin\theta \sin\phi \\ r_3 &= -V_E \sin\beta \cos\chi \\ r_4 &= -V_E \sin\beta \sin\chi \\ r_5 &= V_T \cos\beta \cos\theta. \end{aligned} \tag{18}$$

Thus, we have an interesting physical example in which the non-Abelian geometrical structure to be discussed in the following sections can be explored.

## 4. The $SU(2)$ instanton and two-doublet crossing

Matrix (16) has eigenvalues $R, R, -R, -R$ where $R = (r_1^2+r_2^2+r_3^2+r_4^2+r_5^2)^{\frac{1}{2}}$, so we have the natural generalization of (1) to the case in which the levels with energies $+R$ and $-R$ (which cross at $R = 0$) are each doubly degenerate. When the adiabatically evolving level is itself degenerate, the geometric vector potential becomes a matrix-valued field (non-Abelian potential) [5, 6] defined by

$$A_a^{ij} = \langle\psi_j|\mathrm{i}\partial_a|\psi_i\rangle \tag{19}$$

where $i, j$ run over the labels of the locally single-valued basis in the degenerate space. We proceed to calculate (19) for the problem defined by (16).

One choice of normalized eigenvectors is

$$\psi_1^+ = \frac{\mathrm{i}}{\sqrt{2R(R-r_5)}}\begin{bmatrix} r_3+\mathrm{i}r_4 \\ -r_1+\mathrm{i}r_2 \\ R-r_5 \\ 0 \end{bmatrix} \qquad \psi_2^+ = \frac{\mathrm{i}}{\sqrt{2R(R-r_5)}}\begin{bmatrix} r_1+\mathrm{i}r_2 \\ r_3-\mathrm{i}r_4 \\ 0 \\ R-r_5 \end{bmatrix} \tag{20}$$

corresponding to the eigenvalue $+R$, and

$$\psi_1^- = \frac{\mathrm{i}}{\sqrt{2R(R+r_5)}}\begin{bmatrix} -r_3-\mathrm{i}r_4 \\ r_1-\mathrm{i}r_2 \\ R+r_5 \\ 0 \end{bmatrix} \qquad \psi_2^- = \frac{\mathrm{i}}{\sqrt{2R(R+r_5)}}\begin{bmatrix} -r_1-\mathrm{i}r_2 \\ -r_3+\mathrm{i}r_4 \\ 0 \\ R+r_5 \end{bmatrix}. \tag{21}$$

corresponding to the eigenvalue $-R$. Inserting (20) and (21) into (19) we obtain the potentials

$$A_a^{\pm} = \frac{1}{2R(R\mp r_5)}\begin{bmatrix} r_4\sigma_1 + r_3\sigma_2 - r_2\sigma_3 \\ -r_3\sigma_1 + r_4\sigma_2 + r_1\sigma_3 \\ r_2\sigma_1 - r_1\sigma_2 + r_4\sigma_3 \\ -r_1\sigma_1 - r_2\sigma_2 - r_3\sigma_3 \\ 0 \end{bmatrix} \tag{22}$$

where the first row on the right-hand side of (22) gives the matrix for $A_1^{\pm}$ and so on, ending with $A_5^{\pm} = 0$. We note some similarity with (4) and (8). In the following section we shall see, using a different coordinate system, that the eigenvectors and geometric potentials are in fact independent of $R$—just as, in the $U(1)$ case, the corresponding quantities in the spherical basis were independent of $r$. Thus, our non-Abelian potentials (22) are naturally defined on the sphere $S^4$. To exploit this we project from five dimensions onto the surface of the unit four-dimensional hypersphere via the coordinate transformation

$$r_\mu = \frac{2x_\mu}{1+x^2} \tag{23}$$

$$r_5 = \frac{1-x^2}{1+x^2} \tag{24}$$

where $\mu$ runs from 1 to 4 and $x^2 = x_\mu x_\mu$. We then obtain

$$A_\mu^+ = \frac{1}{2x^2}\begin{bmatrix} x_4\sigma_1 + x_3\sigma_2 - x_2\sigma_3 \\ -x_3\sigma_1 + x_4\sigma_2 + x_1\sigma_3 \\ x_2\sigma_1 - x_1\sigma_2 + x_4\sigma_3 \\ -x_1\sigma_1 - x_2\sigma_2 - x_3\sigma_3 \\ 0 \end{bmatrix} \tag{25}$$

while $A_a^- = x^2 A_a^+$, and $A_5^\pm = 0$.

To show that they are indeed $SU(2)$ instanton potentials, we refer to the paper by Jackiw and Rebbi [11], which discusses the $O(5)$ properties of the instanton. They show that the conventional four-dimensional potentials, $\tilde{A}_\mu$, are related to our $A_\mu$'s by

$$\tilde{A}_\mu = \frac{2}{1+x^2} A_\mu \tag{26}$$

in the present case (note that our $\tilde{}$ notation is the opposite of that in [11]). If we now finally make the coordinate transformation $x_4 \to -x_4$, we find that our $\tilde{A}_\mu^-$ are precisely the negative of the $SU(2)$ instanton fields defined by [11] and [9]:

$$\tfrac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{A}_\mu^{\text{inst}} = \frac{2}{1+x^2}\Sigma_{\mu\nu}x_\nu \tag{27}$$

where $\Sigma_{\mu\nu} = \eta_{i\mu\nu}\sigma_i/2$ with $\eta_{i\mu\nu} = -\eta_{i\nu\mu} = \epsilon_{i\mu\nu}$ for $\mu, \nu = 1, 2, 3$ and $\eta_{i\mu\nu} = \delta_{\mu\nu}$ for $\nu = 4$. The $\tilde{A}_\mu^+$ fields differ from the $\tilde{A}_\mu^-$ fields by a gauge transformation [9].

The demonstration that the geometric vector potentials in the present case are just those of the $SU(2)$ instanton is our main result. However, it is instructive to look at the problem in another way, which casts further light on the geometry.

We recall that in the monopole case, we needed at least two coordinate patches to avoid singularities in the vector potential, and that the potentials were connected at the $S^1$ boundary between the patches by a non-trivial gauge transformation. Indeed, the associated transition function [16, 10] $\exp[\pm \mathrm{i}\phi]$ defines a map from the $S^1$ equator to the $U(1)$ (structure) group, with winding number $\pm 1$ (and similarly for monopoles of higher magnetic charge). In the instanton case, $S^4$ can be covered by two patches with an overlap region which is $S^3$, and the gauge transformation which connects the two corresponding potentials in this $S^3$ provides a map from $S^3$ to $SU(2)$ [10, 12]. These maps are characterized by an integer, the instanton number. This (topological) number is quite analogous to the magnetic charge carried by the monopole, but while the latter is defined via a two-dimensional surface integral of the second-rank field strength tensor, the former involves a four-dimensional surface integral of a fourth-rank tensor, namely $\mathrm{Tr}(F_{\mu\nu}F^*_{\rho\sigma})$, where $F^*$ is the $\epsilon$-dual of $F$.

To bring out the interesting role of $S^3$, and of the patches on $S^4$—and hence to exploit the $U(1)$ monopole analogy further—we now consider our problem using a coordinate system introduced by Yang [12]. He, incidentally, referred to these configuations as generalizations of Dirac's monopole. And in the present case, of course, the configurations are entirely in Euclidean space, and there is no question of interpreting them as tunnelling events in Minkowski space.

## 5. Yang's potentials

Yang uses the following coordinate system:

$$r_i = \frac{2R\xi_i \sin\theta}{1+\xi^2} \qquad i = 1, 2, 3 \tag{28}$$

$$r_4 = \frac{R(1-\xi^2)\sin\theta}{1+\xi^2} \tag{29}$$

$$r_5 = R\cos\theta \tag{30}$$

$$R = (r_i r^i)^{\frac{1}{2}} \tag{31}$$

giving the metric

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2 + \frac{4R^2\sin^2\theta}{(1+\xi^2)^2}\,\mathrm{d}\xi^2. \tag{32}$$

Note that Yang also uses what he calls 'tensor notation', where he ignores the coefficients of the metric, for example he takes the gradient operator in spherical polars as $(\partial_r, \partial_\theta, \partial_\phi)$ rather than $(\partial_r, 1/r\,\partial_\theta, 1/(r\sin\theta)\partial_\phi)$.

Applying an overall sign change (guided by the previous result) and setting $X_j = -\frac{\mathrm{i}}{2}\sigma_j$, Yang's potentials are

$$A_1^\alpha = 0 \tag{33}$$

$$A_2^\alpha = 0 \tag{34}$$

$$A_3^\alpha = \kappa(\tfrac{1}{2}(1+\xi_1^2-\xi_2^2-\xi_3^2)X_1 + (\xi_1\xi_2-\lambda\xi_3)X_2 + (\xi_1\xi_3+\lambda\xi_2)X_3) \tag{35}$$

$$A_4^\alpha = \kappa((\xi_1\xi_2+\lambda\xi_3)X_1 + \tfrac{1}{2}(1-\xi_1^2+\xi_2^2-\xi_3^2)X_2 + (\xi_2\xi_3-\lambda\xi_1)X_3) \tag{36}$$

$$A_5^\alpha = \kappa((\xi_1\xi_3-\lambda\xi_2)X_1 + (\xi_2\xi_3+\lambda\xi_1)X_2 + \tfrac{1}{2}(1-\xi_1^2-\xi_2^2+\xi_3^2)X_3) \tag{37}$$

where $\kappa = 4\mathrm{i}(\mu\cos\theta-\lambda)/(1+\xi^2)^2$. $\mu = +1, \lambda = +1$ corresponds to Yang's region (or coordinate patch) $a$ and $\mu = +1, \lambda = -1$ corresponds to region $b$. Region $a$ includes the 'north pole' $\theta = 0$, and region $b$ includes the 'south pole' $\theta = \pi$. We shall call these regions $N$ and $S$ respectively. A second, gauge-inequivalent field, $A^\beta$, is given in region $N$ by letting $\mu = -1, \lambda = -1$ and in region $S$ by letting $\mu = -1, \lambda = +1$. (This is, as Yang shows, the anti-instanton.)

To obtain these potentials as the geometric vector potentials for our problem we need to rewrite the Cartesian eigenvectors (20) and (21) in terms of Yang's coordinates. Letting $\gamma^\mp = (\mathrm{i}\sqrt{2(1\mp\cos\theta)}(1+\xi^2))^{-1}$ the eigenvectors become

$$\psi_1^\pm = \gamma^\mp \begin{bmatrix} \pm\sin\theta(2\xi_3+\mathrm{i}(1-\xi^2)) \\ \pm 2\sin\theta(-\xi_1+\mathrm{i}\xi_2) \\ (1+\xi^2)(1\mp\cos\theta) \\ 0 \end{bmatrix} \tag{38}$$

$$\psi_2^\pm = \gamma^\mp \begin{bmatrix} \pm 2\sin\theta(\xi_1+\mathrm{i}\xi_2) \\ \pm\sin\theta(2\xi_3-\mathrm{i}(1-\xi^2)) \\ 0 \\ (1+\xi^2)(1\mp\cos\theta) \end{bmatrix} \tag{39}$$

corresponding to the eigenvalues $\pm R$ respectively. Now, using (19) with the index $a$ now running over $\xi_1, \xi_2, \xi_3, \theta$ and $R$, and comparing with Yang's fields (34)–(37) we obtain

$$A_a^+ = A_a^{(\alpha,S)} \tag{40}$$

$$A_a^- = A_a^{(\beta,N)}. \tag{41}$$

The gauge potential $A_a^\alpha$, obtained so far, is defined over only the $S$ coordinate patch, and the potential $A_a^\beta$ over only the $N$ patch. For a full description of the monopole we also need these potentials in the other patches, namely $A_a^{(\alpha,N)}$ and $A_a^{(\beta,S)}$. Gauge potentials in different patches are related by a non-Abelian gauge transformation of the form

$$A_\mu \to A'_\mu = S(x)A_\mu(x)S^{-1}(x) - \frac{\mathrm{i}}{g}(\partial_\mu S(x))S^{-1}(x) \tag{42}$$

where $S$ is an element of the gauge group, in this case $SU(2)$.

In the present case, we may associate a gauge transformation of the potentials with a unitary transformation, $\Lambda$, applied to the basis vectors in each degenerate subspace:

$$|\psi_i\rangle \to |\psi'_i\rangle = \Lambda_{ij}|\psi_j\rangle \tag{43}$$

$$A_a^{ij} \to A_a^{ij'} = \langle\psi'_j|i\,\partial_a|\psi'_i\rangle \tag{44}$$

$$= \Lambda A_a \Lambda^{-1} + \mathrm{i}(\partial_a\Lambda)\Lambda^{-1}. \tag{45}$$

Since the intersecting Kramers doublets do indeed describe the $SU(2)$ instanton, we expect that the other potentials, $A_a^{(\alpha,N)}$ and $A_a^{(\beta,S)}$, should arise from a different choice of basis vectors.

Both $A_a^{(\alpha,S)}$ and $A_a^{(\beta,N)}$ are gauge transformed to their other patch counterparts by (42) with [12]

$$S = (1 - \xi^2 + 2\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{\sigma})/(1+\xi^2). \tag{46}$$

Thus, we apply the basis change $\Lambda = S$ to the basis vectors $|\psi_i^\pm\rangle$ to obtain an alternative basis set

$$\psi_1^{'\pm} = \gamma^\mp \begin{bmatrix} \mp\sin\theta(1+\xi^2) \\ 0 \\ \mathrm{i}(1-\xi^2+2\mathrm{i}\xi_3)(1\mp\cos\theta) \\ -2(1\mp\cos\theta)(\xi_1 - \mathrm{i}\xi_2) \end{bmatrix} \tag{47}$$

$$\psi_2^{'\pm} = \gamma^\mp \begin{bmatrix} 0 \\ \pm\sin\theta(1+\xi^2) \\ -2(\xi_1+\mathrm{i}\xi_2)(1\mp\cos\theta) \\ \mathrm{i}(1-\xi^2-2\mathrm{i}\xi_3)(1\mp\cos\theta) \end{bmatrix} \tag{48}$$

using the previous definition of $\gamma^\mp$. By putting these new vectors into (19) they yield $A_a^{'+} = A_a^{(\alpha,N)}$ and $A_a^{'-} = A_a^{(\beta,S)}$.

Thus, we have identified the two geometric potentials associated with the higher and lower energy Kramers doublets exactly (up to a gauge transformation) with Yang's two gauge-inequivalent $SU(2)$ generalized monopoles:

$$A_a^+ = A_a^\alpha \tag{49}$$

$$A_a^- = A_a^\beta. \tag{50}$$

Yang shows explicitly that these instanton fields minimize the four-dimensional Euclidean Yang–Mills action [9]. He also remarks that since he has proved that his fields $\alpha$ and $\beta$ are the only $SO(5)$ symmetrical $SU(2)$ gauge fields (other than the trivial case), and since the $SU(2)$ instanton is $SO(5)$ symmetrical when conformally mapped to $S^4$ [11], the latter must be identical with one of his fields $\alpha$, $\beta$ (the anti-instanton corresponding to the other). We have verified this identity of fields explicitly by calculating the geometric vector potential associated with the adiabatic evolution of two Kramers doublets.

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After acceptance of this paper we received a note from Péter Lévay informing us of his work [17, 18] which extends that of [7] using a quaternionic formalism, and anticipates some of our results. We thank Dr Lévy for drawing our attention to this work, and apologise for omitting reference to it.

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